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Collinearity, convergence and cancelling infrared divergences

Martin Lavelle and David McMullan
School of Mathematics and Statistics, University of Plymouth
Plymouth, PL4 8AA, U.K.
E-mail: mlavelle@plymouth.ac.uk dmcmullan@plymouth.ac.uk

Abstract: The Lee-Nauenberg theorem is a fundamental quantum mechanical result which provides the standard theoretical response to the problem of collinear and infrared divergences. Its argument, that the divergences due to massless charged particles can be removed by summing over degenerate states, has been successfully applied to systems with final state degeneracies such as LEP processes. If there are massless particles in both the initial and final states, as will be the case at the LHC, the theorem requires the incorporation of disconnected diagrams which produce connected interference effects at the level of the cross-section. However, this aspect of the theory has never been fully tested in the calculation of a cross-section. We show through explicit examples that in such cases the theorem introduces a divergent series of diagrams and hence fails to cancel the infrared divergences. It is also demonstrated that the widespread practice of treating soft infrared divergences by the Bloch-Nordsieck method and handling collinear divergences by the Lee-Nauenberg method is not consistent in such cases.

Keywords: QCD, Standard Model, Gauge Symmetry, Renormalization Regularization and Renormalons.

Contents

1. Introduction

Gauge theories have very good ultraviolet properties [1] and, building upon this, have become the bed rock for the standard model of particle physics. In stark contrast to this, though, they have very bad infrared behaviour which obstructs a deeper understanding of their large scale properties and hence their predictive power.

The infrared problem in gauge theories has a long history. It was addressed, and to a limited sense solved, in one of the earliest papers on QED [2]. In that paper, Bloch and Nordsieck recognised that in, for example, Coulombic scattering there is always the possibility for the emission of a soft photon no matter how far the electron is from the scattering event. The fundamental reason for this is that the photon is massless, so there is always enough energy to emit a photon with a suitably long wavelength. Indeed, given the finite energy resolution of any experiment, the processes of electron scattering and electron scattering accompanied by the emission of an arbitrary number of soft photons (so that overall their energy is less than the experimental resolution) are indistinguishable. The inclusive cross-section formed by summing over all soft final photons was shown, by Bloch and Nordsieck, to be infrared finite but resolution dependent.

The Bloch-Nordsieck mechanism for dealing with the infrared problem in QED is physically appealing but not fully satisfying theoretically. It did not explain why the virtual corrections to tree-level Coulombic scattering suffered from infrared divergences in

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the first place, but only that they could be cancelled against the processes of emitting real but soft photons. There is also an unattractive time asymmetry in the Bloch-Nordsieck argument: one is forced to assume that the in-state is *just* an electron while the out-state has an electron *and* the experimentally unobservable soft photons. Surely, though, the in-state could also be contaminated with unobservable photons? However, including these would introduce new soft divergences and it is not immediately clear how they could be cancelled.

In the early sixties our understanding of the infrared properties of QED was greatly advanced by the works of Lee and Nauenberg [3] and Chung [4]. These authors addressed different questions concerning the infrared and hence had very little overlap in their conclusions. However, their insights highlighted complementary aspects of the infrared that, we feel, must ultimately be fused into a common strategy for dealing with the large scale aspects of gauge theories.

On the face of it, the most conservative approach was that of Lee and Nauenberg. They wanted to extend the Bloch-Nordsieck mechanism to include the additional (collinear or mass) infrared structures that arise when, for example, the mass of the electron is taken to zero in Coulombic scattering. Building upon lessons learnt through explicit calculations by Kinoshita [5], they were able to prove a rather general quantum mechanical result. When applied to a field theory with massless fields, their formal argument concludes that cross-sections are free of both soft and collinear divergences if summed over both final *and* initial degenerate states. Degenerate means here degenerate in energy up to the resolving power of any given experiment. It is important to note, though, that they do not explicitly investigate how soft divergences cancel in this way and assume "*the infrared divergence has already been eliminated by including the contributions due to the emissions of soft photons*" (see the discussion following equation (20) in [3]).

Chung, on the other hand, was not concerned with additional infrared structures. He wanted to go beyond the cross-section approach of Bloch-Nordsieck and develop an infrared finite S-matrix description of scattering in QED. The challenge he faced was that the virtual processes introduced an infrared divergence at the S-matrix level, while the emission of a soft photon only produces an infrared contribution when integrated over in the cross-section. His response to this was to introduce a new type of process: the emission and absorption of a very specific coherent state of photons. These photons were built up of wave-packets smeared by functions that contained the necessary infrared singularities to cancel the virtual contribution. Note that *both* emission and absorption are essential here, as is the inclusion of the disconnected process where the photon does not interact at all with the electrons (see [4, figure 2]). The fact that these enter at the same order of perturbation theory as the virtual process clearly illustrates the non-standard (coupling dependent) nature of the coherent state wave packets.

Chung also talks about a resolution but now it refers to the range in momenta that the infrared coherent state wave packets are integrated over. Its role is to provide a divide between hard and soft processes, and to this extent it is arbitrary. Through it, though, one could argue that the coherent states ameliorated the infrared while not altering the ultraviolet properties of the theory.

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The coherent state approach was considerably refined by Kibble [6] and then later by Kulish and Faddeev [7]. These works showed that the origin of the soft infrared divergences could be traced back to the fact that the asymptotic interaction in QED never vanishes due to the masslessness of the photon. An asymptotic interaction picture could be constructed, resulting in a theory of infra-particle scattering. The infra-particle being identified as a distorted free particle, where the field dependent distortion can be identified with Chung's infrared coherent state. A characteristic feature of this approach is the abandonment of a particle description. That is, the charges are not identified with poles in the propagator, but rather with branch cuts.

The non-trivial asymptotic dynamics associated with soft degrees of freedom has been extended to include collinear structures [8, 9], at least to the lowest non-trivial order in the coupling. It has also been formally extended to include some aspects of the much more complicated infrared properties found in non-abelian theories [10]. Through this we see that the S-matrix approach to the infrared problems, based on the recognition that massless particles imply a non-trivial asymptotic dynamics, provides a consistent formalism for addressing some key issues related to both soft and collinear divergences. However, as it stands, this approach is unattractive computationally and physically since, in particular, it relies on the use of old fashioned time-ordered perturbation theory and the loss of a particle description of charges [7].

The Lee-Nauenberg approach to the infrared seems, at first sight, less of a departure from the more conventional techniques used in particle physics. Certainly, the cancellation of collinear divergences in Coulombic scattering [3] or pair production (see, for example, [11, section 8.6]) is more familiar and simpler to calculate in the cross-sectional approach (compare with [8] and [9]). This has made it a much more attractive way to understand the infrared for the bulk of the particle physics community. To quote Sterman (see page 443 of [12]): "*For applications to high-energy scattering, its importance has thus far been more conceptual than practical, but it is a fundamental theorem of quantum mechanics and puts many specific results in perspective.*" Given this central role that it plays in our understanding of the infrared, it is important that we have a proper understanding of how to use the Lee-Nauenberg theorem in quantum field theory.

In this paper we want to see, through concrete examples, how the Lee-Nauenberg theorem should be used in practice in field theory. In their original paper, Lee and Nauenberg used, at least from a modern point of view, some non-standard techniques that obscure a full understanding of their method and mask some important consequences of this approach to the infrared. Although several authors [13 – 15] have revisited some aspects of the arguments used by Lee and Nauenberg, there has not been, to the best of our knowledge, a systematic reappraisal of how their method should be applied to gauge theories when there are both initial and final state degeneracies. Given the relevance of precisely this type of process to the forthcoming LHC era in particle physics, such a reassessment of the role of the Lee-Nauenberg theorem is, we feel, particularly timely.

In the original paper [3], two examples were presented to illustrate their mechanism for collinear cancellations. Their work was restricted to the first few degenerate diagrams that arise in these processes. In this paper we plan to present these two examples again,

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but now in full. That is, we will not arbitrarily truncate the Lee-Nauenberg method, but allow for the full degeneracy required in their theorem. The conclusions of this will be quite striking: we will see that there is an incompatibility in these processes between the Bloch-Nordsieck approach to the soft infrared and the Lee-Nauenberg treatment of the collinear regime. We will then explain how a mass-resummation is needed to correctly implement the Lee-Nauenberg proposal, a point not seen in their original presentation. Finally, we will expose a problem with the convergence of the expected infrared cancellation claimed by the theorem when there are final and initial state degeneracies.

In section 2 we discuss Coulombic scattering as the mass of the electron becomes small. This example was the main application considered in [3]. We include it here to both fix notation and to point out an inconsistency in the way soft and collinear divergences were dealt with by Lee and Nauenberg. In section 3 we will treat the example of Coulombic scattering accompanied by an observed out-going jet of collinear photons. Here we will consider, as Lee-Nauenberg did, one out-going photon collinear with the in-coming electron. We will see why a mass resummation is needed in order for Lee-Nauenberg's claims to be implemented and then start to explore the role of disconnected processes in the cancellation of collinear divergences. In section 4 we will allow for an arbitrary number of photons in the final jet. The combinatorics that arises will be clarified and we will see how collinear divergences are meant to cancel for such full jets. The techniques developed will then, in section 5, be applied to soft divergences and we will identify the mechanism for soft cancellation in the Lee-Nauenberg approach. Finally, in section 6, we will focus on the thorny issue of the convergence of the various series in terms of the number of photons. We will see that the series are, in fact, divergent. The cancellation of collinear or soft divergences only arises, as will be discussed, for a very specific ordering of the respective series. We will conclude with a list of open problems related to the implementation of the Lee-Nauenberg theorem in quantum field theory.

2. Degeneracies in Coulomb scattering

Coulombic scattering of an electron is a basic process in field theory and is the main example discussed by Lee and Nauenberg. We will follow their presentation by first considering the scattering of massive electrons and then investigate the high energy limit where the mass, m, can effectively be taken to zero.

To this end we consider the process where an in-coming electron of momentum p is scattered off a nucleus and becomes an out-going electron with momentum p'. We will always work in the lab frame, where the nucleus is taken to be static, unless otherwise stated.

The tree-level cross-section, as described in figure 1, is given by

$$\frac{d\sigma_0}{d\Omega} = \frac{\alpha^2}{|\underline{q}|^4} |\bar{u}'\gamma_0 u|^2, \tag{2.1}$$

where we write $\bar{u}' = \bar{u}(p')$ and $u = u(p)$, and there is an implied sum over final spins and average over initial spins. Note that, in order to keep track of in-coming and out-going particles, we adopt the convention in this paper that all out-going particle momenta are primed.

As is well known, after the usual ultraviolet renormalisation, the one-loop correction to this process contains two terms characterised by the structure functions F_1 and F_2. The function F_2 is responsible for the anomalous magnetic moment of the electron and is infrared safe, so we will neglect it. The other structure function is much more important to us and results in the replacement of γ_0 in the tree-level S-matrix by $\gamma_0(1+F_1)$. Thus at the cross-section level we have

$$\frac{d\sigma_0}{d\Omega} \to \frac{d\sigma_0}{d\Omega}(1+2F_1)\,, \tag{2.2}$$

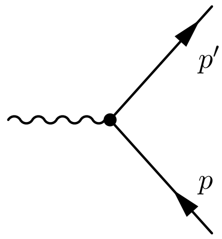

Figure 1: Coulombic scattering at tree-level.

where the factor of 2 arises from the two cross terms that contribute at this order in the coupling. The key point to note is that F_1 is singular in the infrared. Using dimensional regularisation in $D = 4+2\varepsilon_{\rm IR}$ dimensions to regulate the soft infrared divergences, we have

$$2F_1 = \frac{e^2}{4\pi^2}\left[-\frac{1}{\hat\varepsilon_{\rm IR}}\left(\ln\left(\frac{Q^2}{m^2}\right)-1\right)+\frac{1}{2}\ln^2\left(\frac{Q^2}{m^2}\right)+\frac{1}{2}\ln\left(\frac{Q^2}{m^2}\right)\left(1-2\ln\left(\frac{Q^2}{\mu^2}\right)\right)\right]\,, \tag{2.3}$$

where $Q^2 = -(p'-p)^2$, $1/\hat\varepsilon = 1/\varepsilon+\gamma-\ln 4\pi$ and μ is the mass scale that enters dimensional regularisation. In this expression for F_1 we have neglected any terms that are finite as either $\varepsilon_{\rm IR}\to 0$ or $m\to 0$.

The soft infrared divergence can be eliminated using the Bloch-Nordsieck argument by including the degenerate process of soft emission by photons where the photon energy is less than the energy resolution Δ of the detector.

Let us recall how this soft infrared cancellation works in practice. The two diagrams shown in figure 2 contribute to this process at this order in perturbation theory. Note that for this soft emission we can take both p' and k' to be on-shell. The overall out-going momentum is then $p'+k'$ which, as k' is soft, we take to be degenerate with the tree-level process.

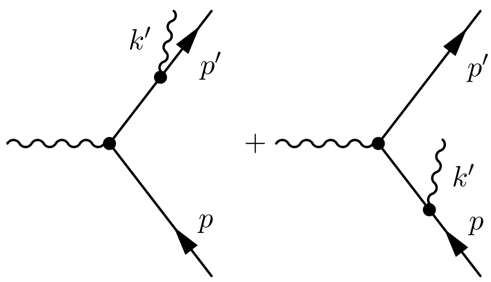

Figure 2: Soft emission from the out-going and in-coming electron.

The S-matrix for this is

$$-ie^2\bar u'\left(\frac{2p'{\cdot}\epsilon'+\not\epsilon'\not k'}{2p'{\cdot}k'}\gamma_0-\gamma_0\frac{2p{\cdot}\epsilon'-\not k'\not\epsilon'}{2p{\cdot}k'}\right)u\,, \tag{2.4}$$

where $\epsilon' = \epsilon(k',\lambda')$, with λ' being the polarisation label for the photon. Now, to simplify the extraction of the soft infrared divergence, we are free to drop the $\not k'$ terms in the numerator and hence arrive at the familiar eikonal expression for the S-matrix elements:

$$-ie^2\left(\frac{p'{\cdot}\epsilon'}{p'{\cdot}k'}-\frac{p{\cdot}\epsilon'}{p{\cdot}k'}\right)\bar u'\gamma_0 u\,. \tag{2.5}$$

Here we recognise the conserved current

$$J^\mu(k') = \frac{p'^\mu}{p'\cdot k'} - \frac{p^\mu}{p\cdot k'}\,, \tag{2.6}$$

associated with the classical change in momentum of the charge. In the cross-section we need to sum over the photon's polarisations and, due to the gauge invariance of the current (2.6), this results in the replacement of $\epsilon'^\mu\epsilon'^\nu$ by $-g^{\mu\nu}$. Hence, by integrating over the physically undetectable soft photons, we arrive at the inclusive cross-section

$$\frac{d\sigma_0}{d\Omega}(1+2F_1+R_s)\,, \tag{2.7}$$

where, up to soft and collinear finite terms, the contribution of real soft emitted photons is

$$\begin{aligned} R_s &= e^2\int_{\text{soft}}\frac{d^{D-1}k'}{(2\pi)^2 2\omega'}\left(\frac{2p'\cdot p}{p'\cdot k' p\cdot k'} - \frac{p'^2}{(p'\cdot k')^2} - \frac{p^2}{(p\cdot k')^2}\right) \\ &= \frac{e^2}{4\pi^2}\left[\frac{1}{\hat\varepsilon_{\text{IR}}}\left(\ln\left(\frac{Q^2}{m^2}\right)-1\right) - \frac{1}{2}\ln^2\left(\frac{Q^2}{m^2}\right) + \ln\left(\frac{Q^2}{m^2}\right)\left(1+2\ln 2+\ln\left(\frac{\tilde\Delta^2}{\mu^2}\right)\right)\right]. \end{aligned} \tag{2.8}$$

Comparing this with (2.3), we see that the inclusion of this degenerate process cancels the soft divergences and also the double logs of the collinear, mass singularities. We are thus left with the collinearly divergent terms

$$\frac{e^2}{\pi^2}\ln\left(\frac{Q}{m}\right)\left[\frac{3}{4}+\ln 2 - \ln\left(\frac{Q}{\tilde\Delta}\right)\right]. \tag{2.9}$$

Note that in arriving at this expression we have worked in the Breit frame. Hence it is the resolution $\tilde\Delta$ in that frame that arises in (2.9). To translate back into the lab frame we note that in the Breit frame

$$Q = 2\tilde E\sqrt{1-\frac{m^2}{\tilde E}}\,, \tag{2.10}$$

where $\tilde E$ is the electron's energy in the Breit frame. Hence, since

$$\frac{\tilde E}{\tilde\Delta} = \frac{E\sin(\frac{1}{2}\phi)}{\Delta\sin(\frac{1}{2}\phi)} = \frac{E}{\Delta}\,, \tag{2.11}$$

where ϕ is the scattering angle for the electrons in the lab frame, we see that the residual collinear divergence describe by (2.9) contributes to the cross-section as

$$\frac{e^2}{\pi^2}\ln\left(\frac{E}{m}\right)\left[\frac{3}{4} - \ln\left(\frac{E}{\Delta}\right)\right]\frac{d\sigma_0}{d\Omega}\,, \tag{2.12}$$

where we have dropped collinear finite terms. Here we see the main content of the Bloch-Nordsieck method: by summing over out-going processes that are degenerate in energy with the scattering of massive electrons, we arrive at an infrared finite but resolution dependent cross-section.

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The masslessness of the photon also means that one could have non-detectable incoming soft photons. They are not included in the Bloch-Nordsieck approach but one could easily construct an alternative procedure where one summed over initial photons that were absorbed by the electron and neglected the out-going photons. This would also be infrared finite if the ensemble of initial photons had the same distribution as the out-going ones. However, it is not so obvious how including both initial *and* final soft photons would lead to an infrared finite result. Although, in principle, covered by the Lee-Nauenberg theorem, this point is not addressed by [3], so we shall postpone a detailed discussion of this until later and press on with the collinear divergence in equation (2.12) that arise as $m \to 0$.

In the high energy limit, the mass of the electron becomes negligible and we have a new class of degenerate processes where the emitted photon can be nearly parallel to the out-going electron. In this collinear configuration, given a finite angular resolution δ in our detector, we can only measure the total out-going energy E. We do not know how it is distributed between the electron and any degenerate photons. Given that we have already integrated over soft photons, this means that we should include collinear photons with energy from the resolution Δ to the total energy E. We call these semi-hard collinear photons.

The emission of a photon collinear with the out-going electron can take place from either electron line. However, it only forces an internal line to go on-shell, and potentially causes an infrared divergence, if it is emitted from the out-going electron. So we only need to consider the process where the photon of momentum k' is emitted from the out-going electron that ends up with momentum p_1', where $p_1' + k' = p'$. For semi-hard collinear photons, this means that k' and p_1' are on-shell, but p' is not. We denote the energy of the out-going electron by E_1 where $E_1 = E - \omega'$. The S-matrix for this can be read off directly from figure 3 to give

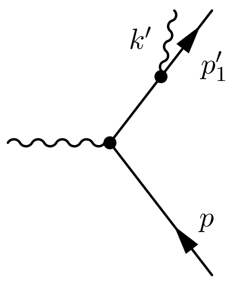

Figure 3: Collinear emission from the out-going electron.

$$- i\frac{e^2}{2p_1'\cdot k'}\bar{u_1}'(2p_1'\cdot\epsilon' + \not\epsilon'\not k')\gamma_0 u\,, \tag{2.13}$$

where now $\bar{u_1}' = \bar{u}(p_1')$.

This contributes to the cross-section the term

$$\frac{e^4}{(p_1'\cdot k')^2}\left[4(p_1'\cdot\epsilon')^2(p_1'\cdot\tilde{p}) - 4(p_1'\cdot\epsilon')(\tilde{p}\cdot\epsilon')(p_1'\cdot k') - 2(\epsilon'\cdot\epsilon')(\tilde{p}\cdot k')(p_1'\cdot k')\right]\,, \tag{2.14}$$

where $\tilde{p}$ is defined by $\tilde{\not p} = \gamma_0 \not p \gamma_0$ so that $4\tilde{p}\cdot p' = |\bar{u}'\gamma_0 u|^2$. Prior to integrating over the semi-hard jet of photons, we need to sum over the photon polarisations. In contrast to the situation that arises in the soft case where we could simply replace $\epsilon_\mu'\epsilon_\nu'$ by $-g_{\mu\nu}$, we now need to use the more general identity that

$$\sum_{\lambda'}\epsilon_\mu'\epsilon_\nu' = -g_{\mu\nu} - \frac{k_\mu' k_\nu'}{\omega'^2} + \frac{k_\mu'\eta_\nu + k_\nu'\eta_\mu}{\omega'}\,, \tag{2.15}$$

with η being the unit time-like vector so that, e.g., $\eta{\cdot}p' = \eta{\cdot}p = E$. Note that in [3] the calculation used an explicit helicity basis for the spinors. Here we will see that the use of (2.15) simplifies the calculations.

After summing over polarisations in (2.14), we get

$$\frac{e^4}{p_1'{\cdot}k'}\left[\left(1+\frac{2E_1}{\omega'}\right)4(p_1'{\cdot}\tilde{p})+\left(1+\frac{E_1}{\omega'}\right)4(k'{\cdot}\tilde{p})\right], \tag{2.16}$$

where we have dropped all collinear finite terms. As discussed in appendix B, for collinear photons and electrons we can, up to collinear finite terms, write

$$k'{\cdot}\tilde{p} = p'{\cdot}\tilde{p}\,\frac{\omega'}{E} \qquad \text{and} \qquad p_1'{\cdot}\tilde{p} = p'{\cdot}\tilde{p}\,\frac{E_1}{E}\,. \tag{2.17}$$

Hence (2.16) becomes

$$e^4|\bar{u}'\gamma_0 u|^2\frac{E_1^2+E^2}{(p_1'{\cdot}k')E\omega'}\,. \tag{2.18}$$

We can now use this to build up the inclusive cross-section for the emission of collinear photons. In doing this it is important to note that, since the emitted photon is now not soft, the in-coming and out-going electrons have different energies. This means that in the cross-section we must include the energy weighting E_1/E familiar from the Bethe-Heitler description of bremsstrahlung (see, for example, [16, section 5-2-4], or the discussion of energy weighting on page 499 of [11]). The resulting cross-section is given by

$$e^4|\bar{u}'\gamma_0 u|^2\int\limits_{\substack{\text{semi-hard}\\\text{cone}}}\frac{d^3k'}{(2\pi)^3 2\omega'}\frac{E_1^2+E^2}{(p_1'{\cdot}k')E\omega'}\frac{E_1}{E}\,, \tag{2.19}$$

where we write

$$p_1'{\cdot}k' = \tfrac{1}{2}\omega' E_1\left(\theta_1^2+\frac{m^2}{E_1^2}\right), \tag{2.20}$$

and θ_1 is the (small) angle between the out-going electron and photon in the lab frame.

Performing the angular integration over the cone with opening angle δ, and dropping collinear finite terms, we get

$$\frac{e^4}{4\pi^2}|\bar{u}'\gamma_0 u|^2\ln\left(\frac{E\delta}{m}\right)\frac{1}{E^2}\int_\Delta^E\frac{E_1^2+E^2}{\omega'}d\omega'\,. \tag{2.21}$$

This final integral can be readily evaluated to yield the cross-section for semi-hard collinear emission:

$$-\frac{1}{2}\frac{e^2}{\pi^2}\ln\left(\frac{E\delta}{m}\right)\left[\frac{3}{4}-\ln\left(\frac{E}{\Delta}\right)-\frac{\Delta}{E}+\frac{1}{4}\frac{\Delta^2}{E^2}\right]\frac{d\sigma_0}{d\Omega}\,. \tag{2.22}$$

Comparing this result with (2.12) we see that just including the out-going semi-hard collinear photons does not completely remove the residual collinear divergences. In particular, the pre-factor of a half found in (2.22) obstructs the cancellation of the divergent terms found in the Bloch-Nordsieck result (2.12). However, following Lee-Nauenberg, if we now include the in-coming degenerate process whereby a semi-hard photon is absorbed by the in-coming electron, then we will get another contribution equal to (2.22) and we thus see the cancellation of the mass logarithms in (2.12). This was the conclusion reached by Lee and Nauenberg following their equation (21).

However, there are still collinear divergent terms in (2.22) that are linear and quadratic in Δ/E and these must be cancelled in order to get a collinear finite cross-section. To trace what is going on here, we note that these terms come from the semi-hard energy integral in (2.21). Indeed the energy integral splits into two terms:

$$\int_{\Delta}^{E} \frac{E_1^2+E^2}{\omega'} d\omega' = 2E^2 \int_{\Delta}^{E} \frac{d\omega'}{\omega'} + \int_{\Delta}^{E} (\omega' - 2E) d\omega' \,. \tag{2.23}$$

In the first term it is essential that the lower limit of Δ is kept otherwise we would reintroduce the soft divergences. However, the second term is finite as $\Delta \to 0$ and hence we see that what is missing is a soft-collinear contribution from the emitted photon that is finite in the soft regime. Thus we conclude that the separation between soft and semi-hard photons is not a precise division between soft and collinear divergent structures.

It is, in fact, not too difficult to trace where such a term was dropped in the Bloch-Nordsieck mechanism. In the discussion following equation (2.4) we made the normal soft simplification of dropping factors of k' in the numerator. This, however, has thrown away a relevant collinear term. Indeed, precisely the corresponding term in (2.13) generates the divergent terms in (2.22). Reinstating this momentum in (2.5) and integrating the energy from 0 to Δ, we see that (2.12) should be replaced with

$$\frac{e^2}{\pi^2} \ln\left(\frac{E}{m}\right) \left[\frac{3}{4} - \ln\left(\frac{E}{\Delta}\right) + \frac{\Delta}{2E} - \frac{\Delta^2}{8E^2}\right] \frac{d\sigma_0}{d\Omega} \,. \tag{2.24}$$

This represents the Bloch-Nordsieck analysis with all collinear terms retained. Now we see that the consistent combination of the Bloch-Nordsieck treatment of soft photon emission with the emission of semi-hard collinear photons results in the cross-section

$$\frac{1}{2}\frac{e^2}{\pi^2} \ln\left(\frac{E}{m}\right) \left[\frac{3}{4} - \ln\left(\frac{E}{\Delta}\right)\right] \frac{d\sigma_0}{d\Omega} \,. \tag{2.25}$$

As expected, this is still collinearly divergent and we need to include the contribution from initial degenerate states. Now, though, we face a problem not addressed in [3]. We have seen that soft initial states are ignored in the Bloch-Nordsieck analysis, and thus there is no equivalent consistent procedure for including initial soft photon contributions that are not infrared divergent. Hence we are forced to simply add the in-coming version of (2.22) resulting in the cancellation of (2.25) but the retention of the linear and quadratic terms in (2.22). This means that, in such an approach to collinear divergences, we must include in the integral over initial states soft photon contributions which produce collinear divergences but ignore those which generate soft divergences. This is extremely unnatural.

Thus we see that, contrary to the procedure in [3], we cannot in general[1] separately treat the soft and collinear divergences using a mixture of Bloch-Nordsieck and Lee-Nauenberg arguments. This does not mean that the general Lee-Nauenberg theorem is

[1]It should be noted, though, that this problem does not arise if there are only massless particles in the final state since then the soft and collinear structures are dealt with in a consistent manner.

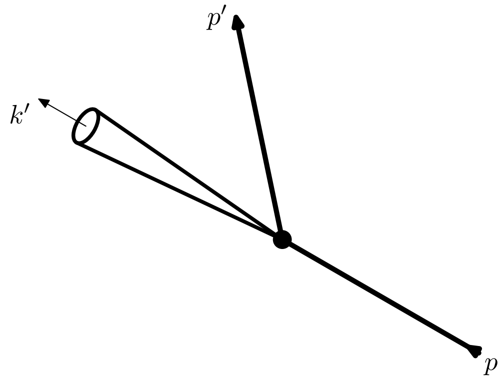

Figure 4: A jet of collinear photons emitted from the incoming electron.

wrong. What is does mean is that we need to understand how to consistently deal with both soft and collinear initial and final state degeneracies. As we have seen, a naive inclusion of absorption from initial soft photons will double the infrared divergences that arise from real processes and hence lose the Bloch-Nordsieck soft cancellation. This suggests that the mechanism for infrared cancellations is more subtle than expected.

In order to understand how we must refine our use of the Lee-Nauenberg method for dealing with the infrared, we now look at the second example discussed briefly in [3, appendix D]. We will now see how this application exposes in a much deeper way the key ingredient of their method when there are both final *and* initial degeneracies.

3. Coulomb scattering accompanied by a collinear jet

In addition to the collinear structures that arise in Coulombic scattering as the mass of the electron becomes negligible, Lee and Nauenberg also considered the process described in figure 4, where we have Coulombic scattering accompanied by the emission of a photon collinear with the incoming electron. The resulting jet of photons of momentum k' in the final state would be clearly distinguished from the outgoing electron as long as a scattering from the target has taken place, such that there is a large (measurable) angle between p' and k', and the energy of the photon is greater than the experimental resolution.

The technical attraction of this process is that the collinear structures that we wish to study already arise at tree-level. This clearly simplifies the identification of the collinear and degenerate aspects of the process. However, even here, the identification of the collinear divergent terms is not straightforward. Here we will present for the first time a comprehensive and more modern approach to this process.

The lowest order diagram that contributes to this is given in figure 5. Its contribution to the S-matrix is:

$$-\frac{ie^2}{(p-k')^2-m^2}\bar{u}'\gamma_0(\not{p}-\not{k}'+m)\not{\epsilon}'u=\frac{ie^2}{2p{\cdot}k'}\bar{u}'\gamma_0(\not{p}-\not{k}'+m)\not{\epsilon}'u\,, \tag{3.1}$$

where p, the incoming electron's momenta, is on-shell.

As in all such processes, infrared divergences arise due to the possibility that an internal line can go on-shell for some choices of the external momenta. Here we see that this occurs when $p{\cdot}k'$ vanishes. Since k' is not soft but the electron's mass is negligible, this happens if the angular separation between k' and p is vanishingly small.

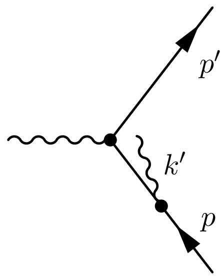

Figure 5: A single collinear photon emitted from the in-coming electron.

We can now identify the associated collinearly divergent contribution to the cross-section associated with this scattering. The resulting cross-section is

$$\frac{2e^4}{(p{\cdot}k')^2}\left\{2(p{\cdot}\epsilon')^2(\tilde{p}'{\cdot}p-\tilde{p}'{\cdot}k')+2(p{\cdot}\epsilon')(\tilde{p}'{\cdot}\epsilon')(p{\cdot}k')-2(\epsilon')^2(\tilde{p}'{\cdot}k')(p{\cdot}k')\right\} \tag{3.2}$$

and we note that here the energy weighting is trivial as the in-coming and out-going electrons both have energy E.

After summing over photon polarisations, we arrive at the cross-section

$$\frac{e^4}{p{\cdot}k'}\left\{\left(\frac{2E}{\omega'}-1\right)4(\tilde{p}'{\cdot}p)+\left(1-\frac{E}{\omega'}\right)4(\tilde{p}'{\cdot}k')\right\}. \tag{3.3}$$

If we now make the collinearity approximations (2.17), we rapidly arrive at the expression

$$P_{0,1}(k',p):=e^4|\bar{u}'\gamma_0 u|^2\frac{2E_1E_0+\omega'^2}{(p{\cdot}k')E\omega'}. \tag{3.4}$$

Here we are modifying the notation introduced in [15] and identifying the Lee-Nauenberg probability $P_{0,1}(k',p)$ as the contribution to the cross-section with no in-coming photons but one out-going photon.

Having identified a collinearly divergent process, we now need to find degenerate processes which also diverge and then sum them up. The most obvious process to consider would be the emission of a photon from the out-going electron that is collinear with the in-coming one, as shown in figure 6.

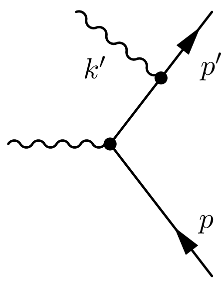

Figure 6: Emission of a photon from the out-going electron.

However, as long as k' is not soft and is parallel to p rather than p', this does not force any internal line in this diagram to go on-shell and hence does not lead to a divergent cross-section. What is needed are degenerate processes where the emitted collinear photon comes from the in-coming electron.

Given that we are working at order e^4 in the coupling, it is not at first obvious that there are any such degenerate processes contributing to the cross-section. However, the Lee-Nauenberg theorem says we need also to consider the initial state degeneracies as described in figure 7. So the in-coming electron itself should be viewed as a degenerate state formed from a mixture of the electron and collinear photons.

For example, we could consider the process whereby an initial photon is first absorbed and then emitted from the in-coming electron or vice-versa as shown in figure 8.

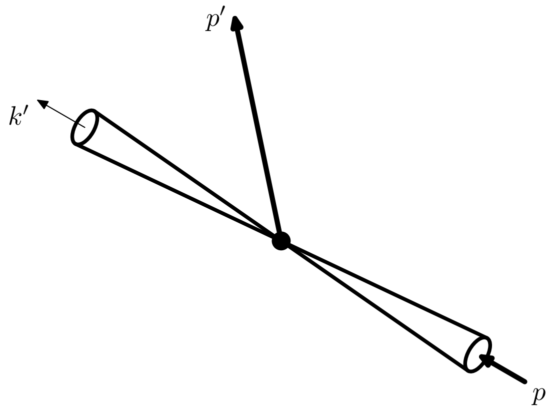

Figure 7: A jet of collinear photons emitted from the incoming charged jet.

In both of these cases, the initial momentum p is distributed between the electron and photon, so $p = p_1 + k$. We stress that p_1 and k are on-shell but p is not. Clearly, collinear divergences will arise if k or k' are parallel to p_1,

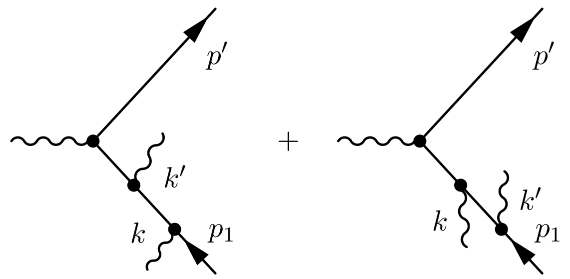

Figure 8: Collinear emission and absorption at lowest order.

The unusual thing about this process is that to contribute to the cross-section at order e^4, we need these emitting and absorbing processes (which are already at order e^3) to interfere with a process of order e. Following [3], we consider the *disconnected* process shown in figure 9 where the in-coming electron is accompanied by a collinear photon that does *not* interact with it.

As the incoming photon is not observed, we should then integrate over all allowed k in the cross-section. The Feynman rule associated with this disconnected process is

$$- e\bar{u}'\gamma_0 u_1 \epsilon'\cdot\epsilon(2\pi)^3 2\omega'\delta^3(k' - k)\,. \tag{3.5}$$

In standard discussions of the S-matrix, disconnected processes like this are ignored as they describe the no scattering situations which can easily be distinguished experimentally. However, if we have both initial and final degeneracies it is not possible to distinguish them. As we will see they produce an important interference with connected diagrams. The immediate effect, though, of the disconnected photon line is to enforce $k = k'$ in the emitting and absorbing processes described in figure 8. This will then put the internal electron line just before the scattering in the diagrams of figure 8 on-shell *for all values of* k'. Clearly this needs to be treated with great care.

How to deal with this is quite subtle and not fully addressed in the literature. Indeed, Lee-Nauenberg handle the issue of vanishing denominators by using a Hamiltonian

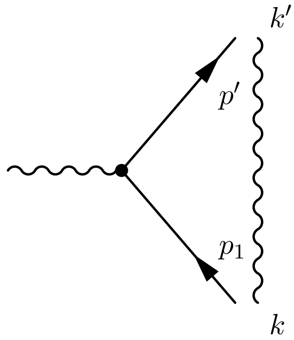

Figure 9: The basic disconnected process.

approach in which, for the first diagram of figure 8, the propagator between the emission and absorption has the form

$$\frac{\not p_1 + \not k + m}{(p_1+k)^2 - m^2 + i(E_1+\omega)\alpha}\,, \tag{3.6}$$

while the propagator before the scattering is

$$\frac{\not p_1 + \not k - \not k' + m}{(p_1+k-k')^2 - m^2 + 2i(E_1+\omega-\omega')\alpha}\,. \tag{3.7}$$

Here we see manifestly the double pole that arises when $k = k'$ and p_1 is on-shell. Also note the factor of 2 in the denominator of (3.7). By retaining the imaginary parts, a vanishing denominator is avoided at the expense of complex parameters. They then extract the real part of the resulting cross-section. This discussion is based on results from "old fashioned" perturbation theory, and does not make clear the need for a mass resummation. We will take a more direct approach to these issues here.

To proceed, we identify the S-matrix for the emission and absorption processes as

$$ie^3 \frac{\bar u'\gamma_0(\not p_1 + \not k - \not k')}{2p_1\!\cdot\!(k-k') - 2k\!\cdot\! k'}\left\{\frac{\not\epsilon'(\not p_1+\not k)\not\epsilon}{2p_1\!\cdot\! k} - \frac{\not\epsilon(\not p_1 + \not k)\not\epsilon'}{2p_1\!\cdot\! k'}\right\}u_1\,, \tag{3.8}$$

where we recall that $u_1 = u(p_1)$ and mass terms in the numerator have been dropped. As discussed above, this will interfere with the disconnected diagram and force $k = k'$. We need to isolate the collinear singularities from those that simply arise from this identification of momenta. To this end, we need to identify those terms in the numerator that vanish as $k \to k'$. As such, it is helpful to write the S-matrix as the sum of two terms:

$$ie^3 \frac{\bar u'\gamma_0 \not p_1}{4(p_1\!\cdot\!(k-k'))(p_1\!\cdot\! k)(p_1\!\cdot\! k')}\left\{(p_1\!\cdot\! k')\not\epsilon'(\not p_1 + \not k)\not\epsilon - (p_1\!\cdot\! k)\not\epsilon(\not p_1 - \not k')\not\epsilon'\right\}u_1 \tag{3.9}$$

and

$$ie^3 \frac{\bar u'\gamma_0(\not k - \not k')}{2(p_1\!\cdot\!(k-k'))(p_1\!\cdot\! k')}\left\{\not\epsilon'\not k'\not\epsilon'\right\}u_1\,. \tag{3.10}$$

Here we have dropped the irrelevant sub-leading term $2k\!\cdot\! k' = -(k-k')^2$ in the denominator and in (3.10) we have, with an eye to the interference with the disconnected diagram, set $k = k'$ in the final braces as the numerator already contains a factor of $k - k'$.

The numerator in (3.9) can be written as (ie^3 times):

$$\begin{aligned}&-4(\epsilon'\cdot\epsilon')(p_1\cdot k')(p_1\cdot k)\bar{u}'\gamma_0 u_1 - 4p_1\cdot(k-k')(p_1\cdot\epsilon)(p_1\cdot\epsilon')\bar{u}'\gamma_0 u_1\\ &+2p_1\cdot(k-k')p_1\cdot(\epsilon'-\epsilon)\bar{u}'\gamma_0\not{\epsilon}\not{k}'u_1 + 2p_1\cdot(k-k')(p_1\cdot\epsilon)\bar{u}'\gamma_0(\not{\epsilon}-\not{\epsilon}')\not{k}'u_1\\ &+2(p_1\cdot k')p_1\cdot(\epsilon'-\epsilon)\bar{u}'\gamma_0\not{\epsilon}(\not{k}'-\not{k})u_1 + 2(p_1\cdot k')(p_1\cdot\epsilon)\bar{u}'\gamma_0(\not{\epsilon}-\not{\epsilon}')(\not{k}'-\not{k})u_1\,.\end{aligned} \tag{3.11}$$

The final four terms are now manifestly of order $(k-k')^2$ and can be disregarded. The first term does not vanish as $k \to k'$ and represents a double pole that arises in that limit. The second term has no double pole and will contribute to the collinear structure of this process.

The upshot of this is that we can write (3.9) in the limit as $k \to k'$ as

$$-ie^3\frac{2(\epsilon'\cdot\epsilon')}{p_1^2-m^2}\bar{u}'\gamma_0 u_1 - ie^3\frac{(p_1\cdot\epsilon')(p_1\cdot\epsilon')}{(p_1\cdot k')(p_1\cdot k')}\bar{u}'\gamma_0 u_1\,, \tag{3.12}$$

plus finite contributions. The first term here has been written in a form that makes clear that it should be interpreted as a mass shift. That such a mass shift should arise in this analysis should come as no surprise since it is precisely these absorption/emission effects with a background that lead to similar divergences when electrons are in an intense laser background [17] or at finite temperature [18]. The method for dealing with these effects through a mass resummation is also now understood [19]. We will assume that such a resummation has taken place and focus on the collinear divergence that arise from the second term in (3.12).

After multiplying by $-2ie\bar{u}_1\gamma_0 u'$ (recall that the factor of two comes from the cross terms that produce the interference), summing over the polarisation λ and integrating over k, the second term in (3.12) yields the following contribution to the cross-section

$$-2e^4\frac{(p_1\cdot\epsilon')(p_1\cdot\epsilon')}{(p_1\cdot k')(p_1\cdot k')}|\bar{u}'\gamma_0 u_1|^2 = -2e^4\frac{(p_1\cdot\epsilon')(p_1\cdot\epsilon')}{(p_1\cdot k')(p_1\cdot k')}|\bar{u}'\gamma_0 u|^2\frac{E_1}{E}\,, \tag{3.13}$$

where we have used the collinear simplification (2.17) and dropped collinear finite terms. If we now sum over polarisations we finally see that (3.9) contributes the collinear divergence

$$-2e^4|\bar{u}'\gamma_0 u|^2\frac{2E_1^2}{(p_1\cdot k')E\omega'}\,. \tag{3.14}$$

The other part of this emmision/absorption process, given by (3.10), contributes to the interference the term

$$-e^4\frac{(\epsilon'\cdot\epsilon')}{p_1\cdot(k-k')(p_1\cdot k')}\mathrm{tr}\,(\not{p}'\not{k}\,\not{k}'\,\not{p}_1)\,. \tag{3.15}$$

The trace can be readily evaluated to give

$$\mathrm{tr}\,(\not{p}'\not{k}\,\not{k}'\,\not{p}_1) = -p_1\cdot(k-k')|\bar{u}'\gamma_0 u|^2\frac{\omega'}{E} + \text{collinear finite terms}\,. \tag{3.16}$$

Hence we see that, after summing over polarisations, (3.10) contributes the collinear divergence

$$-2e^4|\bar{u}'\gamma_0 u|^2\frac{\omega'^2}{(p_1\cdot k')E\omega'}\,. \tag{3.17}$$

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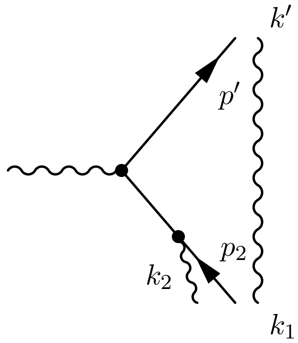


Figure 10: A disconnected process with two in-coming and one out-going collinear photon.

Combining (3.14) and (3.17), and inserting the appropriate energy weighting, we see that the interference contribution to the collinear divergence is

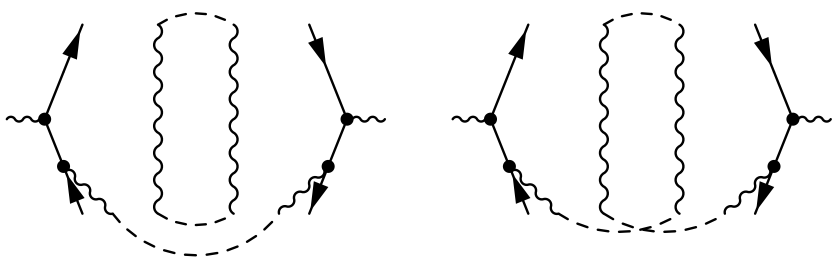
$$P^{\mathrm{c}}_{1,1}(k', p_1) := -2e^4|\bar{u}'\gamma_0 u|^2 \frac{2E_1^2 + \omega'^2}{(p_1\cdot k')E\omega'}\frac{E}{E_1}\,. \tag{3.18}$$

Here the Lee-Nauenberg probability $P^{\mathrm{c}}_{1,1}(k', p_1)$ describes the *connected* interference contribution that arises when there is one in-coming and one out-going collinear photon. (The notation will be explained in full in the next section.) It is clear that this does not cancel (3.4), so we need to look for another degenerate process. The only other collinearly divergent process that has a single photon in its final state is given in figure 10 and arises when there are two in-coming photons, one of which gets absorbed by the in-coming electron. The S-matrix for this is given by

$$\frac{ie^2}{2p_2\cdot k_2}\bar{u}'\gamma_0(\not{p}_2 + \not{k}_2)\not{\epsilon}_2 u_2\, \epsilon'\cdot\epsilon_1 (2\pi)^3 2\omega'\delta^3(k' - k_1)\,, \tag{3.19}$$

where now $u_2 = u(p_2)$ and p_2 is on-shell.

This disconnected process can contribute to the cross-section at order e^4 in two possible ways. If we represent the contraction of photon lines by dashed lines, then either the disconnected photon lines are contracted together to yield a disconnected contribution (as described in figure 11(a)), or one gets the connected contribution described in figure 11(b).



(a) (b)

Figure 11: The two possible contractions of the photon lines.

The disconnected contraction is ignored by Lee-Nauenberg and we will follow their lead on this and postpone any discussion of it until section 6. The connected contraction is easy to calculate once it has been understood that the "loop" of photons can be simply unwound to give a direct contraction of the absorbed photon (see appendix A).

Indeed, the connected contraction is given by

$$-\frac{ie^2}{2p_2\cdot k_1}\bar{u}_2\not\epsilon_1(\not p_2+\not k_1)\gamma_0 u'\,\epsilon'\cdot\epsilon_2(2\pi)^3 2\omega'\delta^3(k'-k_2)\,. \tag{3.20}$$

We now contract (3.19) and (3.20) and integrate over k_1 and k_2 and sum over two of the polarisations. This yields the contribution

$$\frac{e^4}{(2p_2\cdot k')^2}\,\mathrm{tr}\,\left(\not{\bar{p}}'(2p_2\cdot\epsilon'+\not k'\not\epsilon')\not p_2(2p_2\cdot\epsilon'+\not\epsilon'\not k)\right) \tag{3.21}$$

where now $p_2 = p - 2k'$ is on-shell and we still have the final polarisation sum to perform. Note that this is precisely the contribution that would have been obtained from the process of absorbing a collinear photon of momentum k' by an in-coming electron with momentum p_2.

Proceeding as before, we readily arrive at the connected contribution to the cross-section

$$P^{\mathrm{c}}_{2,1}(k',p_2) := e^4|\bar{u}'\gamma_0 u|^2\frac{2E_1E_2+\omega'^2}{(p_2\cdot k')E\omega'}\frac{E}{E_2}\,, \tag{3.22}$$

where the final term is the energy weighting for this process, we recall that $E_1 = E-\omega'$ and so define $E_2 = E - 2\omega'$.

We have so obtained the relevant collinear parts of three degenerate contributions to the process of emitting a collinear jet with one photon in the final state: $P_{0,1}$, $P^{\mathrm{c}}_{1,1}$ and $P^{\mathrm{c}}_{2,1}$. Each of these terms are collinearly divergent when the out-going photon's momenta k' is integrated over a small cone. However, as we shall now show, the sum is finite.

To compare the three expressions (3.4), (3.18) and (3.22) we need to understand the relation between the terms in the denominator. In appendix B we show that we may make the replacement

$$\frac{1}{p\cdot k'} = \frac{1}{p_1\cdot k'}\frac{E}{E_1} = \frac{1}{p_2\cdot k'}\frac{E}{E_2}\,. \tag{3.23}$$

Hence,

$$P_{0,1}+P^{\mathrm{c}}_{1,1}+P^{\mathrm{c}}_{2,1} = 2e^4|\bar{u}'\gamma_0 u|^2\frac{E_1(E+E_2)-2E_1^2}{(p\cdot k')E\omega'}\,. \tag{3.24}$$

The cancellation now follows from the simple identity that $E + E_2 = 2E_1$.

With this result we have reached the end of Lee-Nauenberg's argument. The distinctive feature of this work, which is not stressed in the literature, is the need to incorporate disconnected diagrams. In the introduction to [3], Lee and Nauenberg conclude with the statement that, "*Throughout this paper the question of convergence of the power series is not discussed*". We take this to mean that they recognised the need to incorporate degeneracies where more photons are emitted at the same order in the coupling, i.e., more disconnected photons. We will now turn to this.

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4. Filling out the jet

In the previous section we have seen how collinear divergences cancel in the jet process where a collinear photon is emitted from the in-coming electron. This was achieved by considering all processes degenerate with this single photon final state configuration. The surprising thing about the mechanism for the cancellation was the need to include in the cross-section the connected interference contribution from disconnected S-matrix elements. The analysis was performed at order e^4 in the coupling and is a striking demonstration of the Lee-Nauenberg theorem.

However, having opened the Pandora's box of disconnected diagrams, we see that this is not the end of the story even for this simple process. At the same order in the coupling we can also consider the degenerate processes where there are two or more photons in the final state. As an example, in figure 12 we see a four photon final state process degenerate with the basic one photon event described in figure 6 and occurring at the *same order in perturbation theory*. Clearly, the out-going jet can be filled with an arbitrary number of photons and still be degenerate to the original process. We now need to understand how to deal with these extra divergent processes.

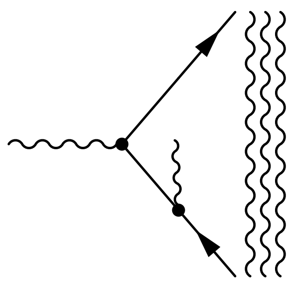
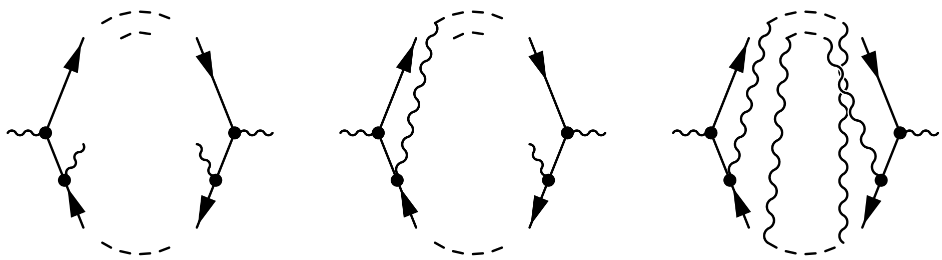


Figure 12: Filling the jet with four photons.

So as not to get lost in the combinatorial details, it is best to first consider the process which we denote $P_{1,2}(k, \{k'_a\}, p_{\text{in}})$, i.e., the Lee-Nauenberg probability corresponding to one in-coming photon with momentum k, two out-going photons with momenta k'_1 and k'_2 and an in-coming electron with momentum p_{in}. These momenta are not arbitrary but satisfy the constraints that $k'_1 + k'_2 = k'$ and $p_{\text{in}} + k = p$. In order to extract the connected component of $P_{1,2}$ we draw the skeleton diagram shown in figure 13(a) where the lower dashed line represents the single in-state contraction and the two upper dashed lines the two out-state contractions. This process comes with a symmetry factor of 1/2! due to the two photon out-states.

We now need to connect the dashed lines by the photon lines in such a way that we produce a connected diagram. Starting with the emitted photon in figure 13(b), we can connect it to one of the out-going contraction lines in two possible ways. This figure



(a) (b) (c)

Figure 13: Building up the connected contribution to $P_{1,2}$.

has now a symmetry factor of $2/2! = 1$ reflecting the two possible contractions. Once this line has been chosen, though, there is then no freedom in completing the rest of the contractions if we are looking for a connected contribution. Thus we arrive at figure 13(c) with a symmetry factor of 1. The momentum delta functions now undo the photon loop by setting $k = k'_1 = k'_2 = k'/2$ and we arrive at an expression equivalent to the original process $P_{0,1}$ but now with momenta $k'/2$ for the out-going photon and $p_{\frac{1}{2}} = p - \frac{1}{2}k'$ for the in-coming electron. Thus we have shown that the Lee-Nauenberg probability $P_{1,2}$ has a connected contribution, $P^{\rm c}_{1,2}$, to the cross-section where

$$P^{\rm c}_{1,2}(k, \{k'_a\}, p_{\rm in}) = P_{0,1}(\tfrac{1}{2}k', p_{\frac{1}{2}})\,. \tag{4.1}$$

This analysis can be readily extended to the process $P_{n-1,n}(\{k_i\}, \{k'_a\}, p_{\rm in})$ where we have $n-1$ in-coming photons, k_i and n out-going photons, k'_a. As before, these momenta are constrained so that:

$$\sum_{a=1}^{n} k'_a = k' \qquad \text{and} \qquad p_{\rm in} + \sum_{i=1}^{n-1} k'_i = p\,. \tag{4.2}$$

To represent this as in figure 13, we would need n dashed lines on top and $n-1$ on the bottom. The symmetry factor for this would be $1/n!(n-1)!$. Connecting the emitted photon with an upper line, as in figure 13(b), can be done in n ways. Keeping the process connected then restricts the next choice to $n-1$ possibilities. Repeating this argument, we have $n!(n-1)!$ contractions leading to the connected process and hence a final symmetry factor of, again, 1. The connected contribution from this is then

$$P^{\rm c}_{n-1,n}(\{k_i\}, \{k'_a\}, p_{\rm in}) = P_{0,1}(\tfrac{1}{n}k', p_{\frac{n-1}{n}})\,. \tag{4.3}$$

In a similar way, we can consider the Lee-Nauenberg probabilities with n in-coming and n out-going photons, $P_{n,n}$ and finally the case with $n+1$ in-coming and n out-going photons $P_{n+1,n}$. What we find is that these have connected contributions related to, respectively, $P^{\rm c}_{1,1}$ and $P^{\rm c}_{2,1}$ which were introduced in section 3. To be precise

$$P_{n,n} = P_{n,n}(\{k_i\}, \{k'_a\}, p_{\rm in})\,, \tag{4.4}$$

where

$$\sum_{a=1}^{n} k'_a = k' \qquad \text{and} \qquad p_{\rm in} + \sum_{i=1}^{n} k'_i = p\,, \tag{4.5}$$

and

$$P_{n+1,n} = P_{n+1,n}(\{k_i\}, \{k'_a\}, p_{\rm in})\,, \tag{4.6}$$

where now

$$\sum_{a=1}^{n} k'_a = k' \qquad \text{and} \qquad p_{\rm in} + \sum_{i=1}^{n+1} k'_i = p\,. \tag{4.7}$$

The connected contributions are then:

$$P^{\rm c}_{n,n}(\{k_i\}, \{k'_a\}, p_{\rm in}) = P^{\rm c}_{1,1}(\tfrac{1}{n}k', p_1)\,, \tag{4.8}$$

and

$$P^{\mathrm{c}}_{n+1,n}(\{k_i\},\{k'_a\},p_{\mathrm{in}}) = P^{\mathrm{c}}_{2,1}\left(\frac{1}{n}k', p_{\frac{n+1}{n}}\right). \tag{4.9}$$

There are no other degenerate processes at this order in perturbation theory.

Having reduced the general connected Lee-Nauenberg probabilities to re-scaled versions of the single photon probabilities calculated earlier, it is now straightforward to evaluate them and we find:

$$P^{\mathrm{c}}_{n-1,n} = \frac{e^4|\bar{u}'\gamma_0 u|^2}{(p{\cdot}k')E\omega'}\left(2n^2E_1E_{\frac{n-1}{n}} + \omega'^2\right); \tag{4.10}$$

$$P^{\mathrm{c}}_{n,n} = -2\frac{e^4|\bar{u}'\gamma_0 u|^2}{(p{\cdot}k')E\omega'}(2n^2E_1^2 + \omega'^2); \tag{4.11}$$

$$P^{\mathrm{c}}_{n+1,n} = \frac{e^4|\bar{u}'\gamma_0 u|^2}{(p{\cdot}k')E\omega'}(2n^2E_1E_{\frac{n+1}{n}} + \omega'^2), \tag{4.12}$$

where now we define $E_m = E - m\omega'$. Note that

$$P^{\mathrm{c}}_{n-1,n} + P^{\mathrm{c}}_{n,n} + P^{\mathrm{c}}_{n+1,n} = \frac{e^4|\bar{u}'\gamma_0 u|^2}{(p{\cdot}k')E\omega'}2n^2(E_1E_{\frac{n-1}{n}} + E_1E_{\frac{n+1}{n}} - 2E_1^2). \tag{4.13}$$

Hence, using the identity that $E_{\frac{n-1}{n}} + E_{\frac{n+1}{n}} = 2E_1$, we see that the sum is zero.

5. Soft divergences revisited

Having seen the essential role played by disconnected processes that produce a connected contribution to the cross-section, we can now revisit the problem left unanswered in section 2 and investigate the cancellation of soft infrared divergences within the Lee-Nauenberg framework.

Recall that, from equation (2.7), we have seen that the infrared divergent content of Coulombic scattering accompanied by the emission of soft photons can be expressed as

$$\frac{d\sigma_0}{d\Omega}\frac{e^2}{4\pi^2}\left(\ln\left(\frac{Q^2}{m^2}\right) - 1\right)\frac{1}{\hat{\varepsilon}_{\mathrm{IR}}}(-1+1) = 0\,. \tag{5.1}$$

Here the -1 term comes from the virtual contribution and the $+1$ from the real emission. Hence the content of the Bloch-Nordsieck approach to soft infrared cancellation can be succinctly summarised by the formula

$$\frac{1}{\hat{\varepsilon}_{\mathrm{IR}}}(-1+1) = 0\,. \tag{5.2}$$

In the Lee-Nauenberg approach we must also consider initial degeneracies and the associated absorption of soft photons as shown in figure 14. As we have seen in other examples of the Lee-Nauenberg method, these are taken to contribute is exactly the same way as the emission process. That is, although it is possible to have a different initial resolution which will change finite terms, the essential infrared divergence will be the same

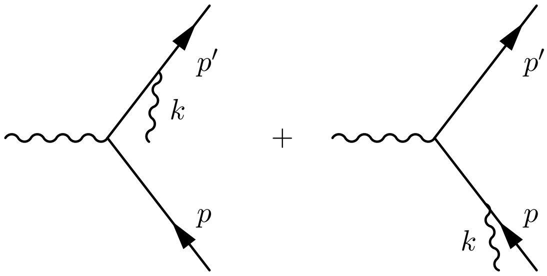


Figure 14: Soft absorption to the out-going and in-coming electron.

if the ensemble of soft photons has the same distribution for both in and out states. Hence, including these initial processes, we find that the soft infrared content of the cross-section becomes

$$\frac{1}{\hat{\varepsilon}_{\text{IR}}}(-1+1+1)\,. \tag{5.3}$$

Clearly this is no longer zero and we have an infrared divergent cross-section.

Of course, we should not stop here. What we have learnt from our analysis of Lee-Nauenberg is that we must include *all* degenerate processes and hence need to consider the processes described in figure 15. Including soft photons compounds the number of degenerate processes as it is now not possible to identify which electron an undetectable soft photon was emitted from or absorbed by.

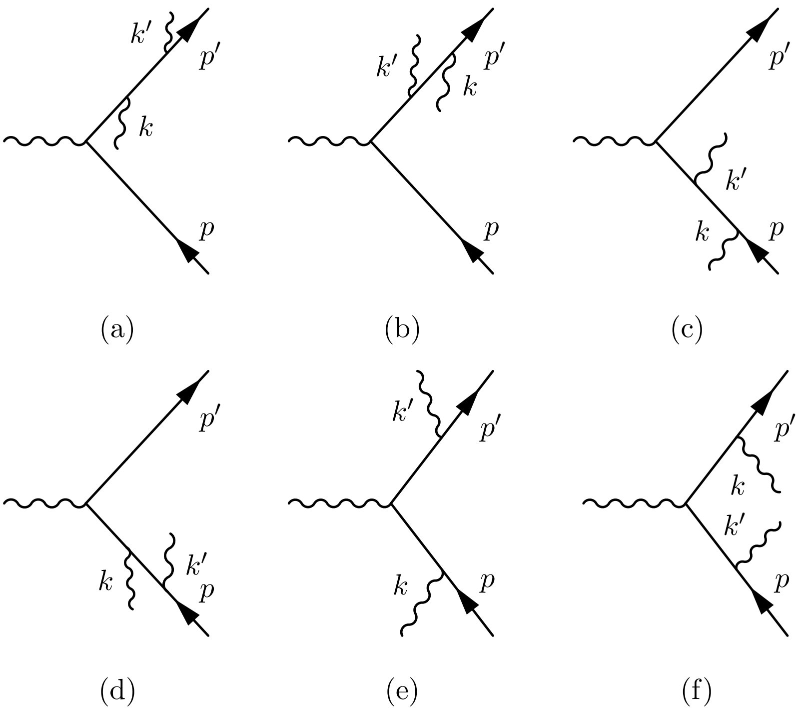


Figure 15: Soft emission and absorption process.

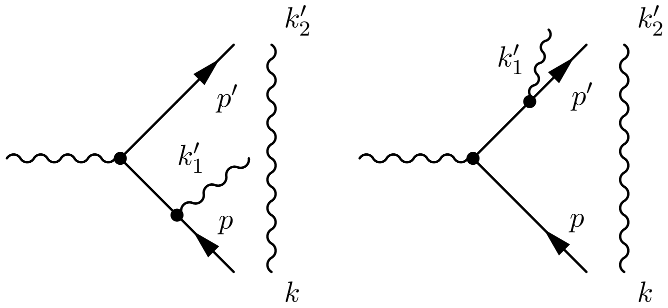


Figure 16: The disconnected processes with one in-coming and two out-going soft photons.

The S-matrix element corresponding to diagrams (a) and (b) in figure 15 is:

$$
\begin{aligned}
&(ie)^3\overline{u}'\not\epsilon'\frac{i}{\not p'+\not k'-m}\not\epsilon\frac{i}{\not p'+\not k'-\not k-m}\gamma_0 u \\
&+(ie)^3\overline{u}'\not\epsilon\frac{i}{\not p'-\not k-m}\not\epsilon'\frac{i}{\not p'+\not k'-\not k-m}\gamma_0 u\,.
\end{aligned} \tag{5.4}
$$

Note that, due to the softness of the photons, here we can take the momentum of the electrons to be just p and p'. Just keeping the leading order soft elements of this (so we assume that the soft mass resummation has been carried out) we get the S-matrix contribution

$$
-ie^3\overline{u}'\gamma_0 u\frac{p'{\cdot}\epsilon'}{p'{\cdot}k'}\frac{p'{\cdot}\epsilon}{p'{\cdot}k}\,. \tag{5.5}
$$

In a similar way, for the processes described in diagrams (c) and (d) of figure 15, we get

$$
-ie^3\overline{u}'\gamma_0 u\frac{p{\cdot}\epsilon'}{p{\cdot}k'}\frac{p{\cdot}\epsilon}{p{\cdot}k}\,. \tag{5.6}
$$

The final two diagrams in figure 15 are slightly different in structure and give the contribution

$$
ie^3\overline{u}'\gamma_0 u\left(\frac{p'{\cdot}\epsilon'}{p'{\cdot}k'}\frac{p{\cdot}\epsilon}{p{\cdot}k}+\frac{p{\cdot}\epsilon'}{p{\cdot}k'}\frac{p'{\cdot}\epsilon}{p'{\cdot}k}\right)\,. \tag{5.7}
$$

Combining all of these absorption/emission processes we get the S-matrix element

$$
-ie^3\overline{u}'\gamma_0 u\left[\left(\frac{p'{\cdot}\epsilon'}{p'{\cdot}k'}-\frac{p{\cdot}\epsilon'}{p{\cdot}k'}\right)\left(\frac{p'{\cdot}\epsilon}{p'{\cdot}k}-\frac{p{\cdot}\epsilon}{p{\cdot}k}\right)\right]\,. \tag{5.8}
$$

This is now contracted with the basic disconnected process described in figure 9 and integrated over the photon momenta, summed over the photon polarisations and multiplied by 2 as it is a cross term in the cross-section. This yields -2 times the emitted or absorbed processes R_s in (2.8). Hence, by including these new degenerate processes, we arrive at the soft infrared content

$$
\frac{1}{\hat{\varepsilon}_{\rm IR}}(-1+1+1-2)\,, \tag{5.9}
$$

to the cross-section. However, we still have a non-vanishing infrared pole.

To proceed, we need to include some more degenerate processes. For example, we could include the process $P_{1,2}$ described in figure 16 (this is the soft version of figure 13(a)). In this we need to sum over all soft photons with the constraint that $0 \le k_1' + k_2' \le \Delta$. Following the process described in figure 13 for extracting the connected contribution to the cross-section, we will arrive at the process described in figure 2 (with k' replaced by k_1') with the constraint that $k_1' = k_2' = k$. Hence the contribution to the cross-section will be as in equation (2.8) but with the integration range up to $\frac{1}{2}\Delta$. However, as we have already discussed, the infrared divergence is insensitive to the value of the resolution and hence the degenerate process $P_{1,2}$ will contribute the same as $P_{0,1}$ to the infrared pole in the cross-section and thus (5.9) becomes

$$\frac{1}{\hat{\varepsilon}_{\mathrm{IR}}}(-1+1+1-2+1)\,. \tag{5.10}$$

At this stage we have recovered an infrared finite cross-section and this is essentially the argument given in [13]. What this seems to show is that the virtual divergence, $P_{0,0}$ is cancelled by the emission $P_{0,1}$ via the Bloch-Nordsieck mechanism, and then the absorption process $P_{1,0}$ is cancelled by the mixed processes $P^{\mathrm{c}}_{1,1}$ and $P^{\mathrm{c}}_{1,2}$. That is,

$$0 = -1+1+1-2+1 = \underbrace{-1+1}_{\text{Bloch}-\text{Nordsieck}} + \underbrace{1-2+1}_{\text{Lee}-\text{Nauenberg}}\,. \tag{5.11}$$

This result seems to reconcile the Bloch-Nordsieck and the Lee-Nauenberg approaches to soft infrared divergences. However, we have so far only included one disconnected photon and, even at this order in perturbation theory, we can and indeed must include an arbitrary number of such photons.

6. The question of convergence

We have seen that, in applying the Lee-Nauenberg theorem to quantum field theory, what one does in practice is to include enough degenerate states to achieve infrared finiteness for the cross-section. For the two jet process discussed in section 2, the *only* relevant degenerate states at order e^4 are those where a photon is emitted from the out-going electron, or absorbed by the in-coming electron. There are *no* disconnected semi-hard contributions. This would also be the case in any process without massless initial charged states.

For the three jet event discussed in section 3, we have seen that disconnected contributions are essential for the cancellation observed in (3.24). We have also just demonstrated in (5.10) that this is the case for soft cancellation if we go beyond Bloch-Nordsieck (which we have seen in section 2 is essential in order to capture the full soft-collinear structure of the theory).

However, why should we stop at these levels of degeneracies? The Lee-Nauenberg theorem requires us to sum over *all* degeneracies. In the three jet system we went beyond the discussion given by Lee and Nauenberg and saw in (4.13) that the Lee-Nauenberg probabilities could be grouped in such a way that the cancellation seems to work for the jet full of disconnected photons. A similar argument can be developed for the soft structures discussed in the last section. Indeed, for soft momenta the construction is greatly simplified since in, for example, (4.3), (4.8) and (4.9), the in-coming electron always has momentum p. The upshot of this is that, as far as the infrared pole is concerned, we have the identities:

$$P^{\rm c}_{n-1,n} = P_{0,1} \qquad P^{\rm c}_{n,n} = P^{\rm c}_{1,1} \quad (n > 0) \quad \text{and} \quad P^{\rm c}_{n+1,n} = P_{1,0}\,, \tag{6.1}$$

along with the relations derived in section 5 that $P_{0,1} = -P_{0,0}$, $P^{\rm c}_{1,1} = -(P_{0,1} + P_{1,0})$ and $P_{0,1} = P_{1,0}$.

The generalisation of (5.10) is then the vanishing of the quantity

$$P_{0,0} + P_{0,1} + \sum_{n=1}^{\infty}(P^{\rm c}_{n,n-1} + P^{\rm c}_{n,n} + P^{\rm c}_{n,n+1})\,. \tag{6.2}$$

Now, however, we need to enquire as to the robustness of this result. That is, we need to ensure that the series involved are converging. Clearly they are not! This is most strikingly see in (6.2), where we can simply absorb the $P_{0,1}$ term into the sum to get the non-vanishing result:

$$P_{0,0} + \sum_{n=1}^{\infty}(P^{\rm c}_{n,n-1} + P^{\rm c}_{n,n} + P^{\rm c}_{n-1,n}) = P_{0,0}\,. \tag{6.3}$$

In fact, this ordering is quite attractive from a coherent state approach or also from the dressing description [21, 20] of gauge invariant charged particles. This is because the virtual term $P_{0,0}$ is already infrared finite in both of those approaches. This would then show that the infrared finiteness is not spoilt by any undetectable soft process. However, expressions like (6.2) or (6.3) cannot be taken seriously as it is clear that the sum is ill defined. Similar conclusions can be reached about the three jet process. A more striking example of how the lack of convergence can lead to unacceptable results is presented in appendix C.

7. Discussion and open problems

We have seen that there are surprising aspects to the application of the Lee-Nauenberg theorem in gauge theories. In particular, one needs to include disconnected diagrams since they can produce connected, interference contributions to cross-sections. These only arise when there are both initial and final state degeneracies. However, they are essential for all soft processes and for collinear multi-jet processes where one of the jets is in the direction of the incoming beam. The apparent cancellations which we have seen for arbitrary numbers of photons are quite remarkable. However, we have also seen that the cancellation is not robust as the expansion in the number of photons is not convergent. We have also demonstrated that there is a mass shift caused by these diagrams which implies the need for a mass resummation in any phenomenological application.

One of the attractive aspects of these calculations is that one can admit the physically intuitive picture that there can be unobservable particles in both the final *and* initial states. The unattractive side to this is the requirement that the degenerate initial and final states have the same ensemble distribution in momentum space. This involves some fine-tuning of how the initial state is prepared which has led to some debate [22, 23] on the preparation of experimental initial states. Reservations on this are also apparent in p. 1554 of [3] and, more recently, in Weinberg's remarks on p. 552 of [24] *"The sum over initial states is more problematic. Presumably one may argue that truly massless particles are always produced as jets accompanied by an ensemble of soft quanta that is uniform within some volume of momentum space. However, to the best of my knowledge no one has given a complete demonstration that the sums of transition rates that are free of infrared divergences are the only ones that are experimentally measurable."*

Another consequence of initial sums is the need for disconnected processes. These in turn produce disconnected contributions to the cross-section which are usually dropped in the literature. A naive treatment of these terms introduces a factor of $\delta^{(3)}(0)$ which should be interpreted as a volume. These ill-defined terms arise because we are using plane wave states. A more careful treatment would use wave-packets concentrated around the beam, so that the beam volume would replace this singularity. Such a description has not yet been constructed.

Although we feel that our analysis has clarified important aspects of the Lee-Nauenberg theorem, we have also seen that, as it stands, it cannot be applied to any soft process or to any collinear process with both initial and final state degeneracies which allows interference from disconnected diagrams. Given the significance of this theorem, we feel that it is essential that the following open questions are answered.

- We have seen that the connected interference terms do not converge for arbitrarily many disconnected photons. A naive attempt at making it converge by using a coherent state normalisation results in a breakdown of the infrared cancellations. Can we find a set of well defined particle states with which the infrared divergences cancel in a well-defined fashion at the level of perturbative cross-sections?

- How can we consistently treat disconnected diagrams? Is there a way to treat them such that the disconnected contributions can be factorised out of the cross-section and hence absorbed into a normalisation? This again requires that the series of Lee-Nauenberg probabilities converges for arbitrarily many disconnected photons.

- The need for summing over initial particles was first recognised as important with the birth of QCD. How then can this analysis be extended to the non-abelian theory? We note that it is essential here to include the soft and collinear effects of three and four gluon vertices.

Although infrared safety allows us to sidestep many of these questions, a deeper understanding of gauge theories will, we feel, follow from answering them.

Acknowledgments

We wish to thank especially Emili Bagan for detailed discussions on many of the topics treated in this paper. We also thank Tom Heinzl, Paul Jameson, Arsen Khvedelidze and Oliver Schröder for discussions.

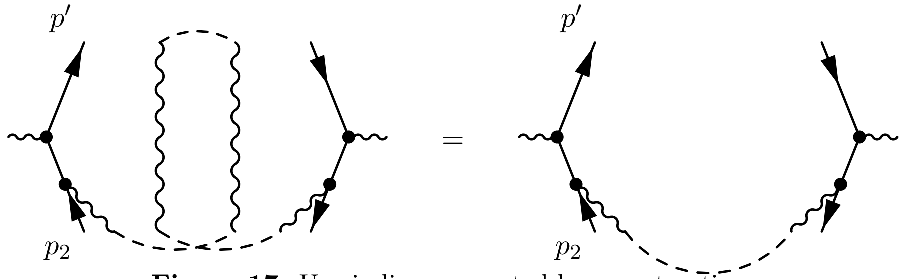


Figure 17: Unwinding connected loop contractions.

A. Contracting photon lines

In many places in this paper we have "unwound" connected lines of photons. Diagrammatically, this can be summarised as the identity shown in figure 17. What we want to do in this appendix is show how this unwinding process follows from the Feynman rule (3.5).

To be concrete, using the momentum assignments in figure 10, the basic S-matrix element is

$$ie^2\frac{\bar{u}'\gamma_0(\not{p}_2+\not{k}_2)\not{\epsilon}_2 u_2}{2p_2\cdot k_2}\epsilon'\cdot\epsilon_1(2\pi)^3 2\omega'\delta^3(k'-k_1)\,. \tag{A.1}$$

In order to have a connected contribution to the cross-section, this must be contracted with (see figure 11(b))

$$-\,ie^2\frac{\bar{u}_2\not{\epsilon}_1(\not{p}_2+\not{k}_1)\gamma_0 u'}{2p_2\cdot k_2}\epsilon'\cdot\epsilon_2(2\pi)^3 2\omega'\delta^3(k'-k_2)\,. \tag{A.2}$$

We recall that in both (A.1) and (A.2) we are using the notation that $\epsilon' = \epsilon(k',\lambda')$, $\epsilon_1 = \epsilon(k_1,\lambda_1)$ and $\epsilon_2 = \epsilon(k_2,\lambda_2)$.

Contracting these two terms together by integrating over the momenta k_1 and k_2 which enforces all the momenta to be k', we arrive at an expression of the form

$$A_{\mu\nu}\sum_{\lambda',\lambda_1,\lambda_2}\epsilon^\mu(\lambda_2)\epsilon^\nu(\lambda_1)\epsilon_\rho(\lambda')\epsilon^\rho(\lambda_1)\epsilon_\tau(\lambda')\epsilon^\tau(\lambda_2)\,, \tag{A.3}$$

where we have suppressed the common k' dependence in the polarisation tensors. Now summing over the polarisations λ_1 and λ_2, and making repeated use of the identity (2.15), allows us to reduce this to

$$A_{\mu\nu}\sum_{\lambda'}\epsilon^\mu(k',\lambda')\epsilon^\nu(k',\lambda')\,. \tag{A.4}$$

This is precisely the contribution that we would have written down for the unwound process in figure 17, where there is just an absorption of a photon of momentum k' by the incoming electron with momentum p_2.

B. Some collinear approximations

At various places in this paper we make approximations based on collinearity. There are two classes of such approximations that we want to discuss in some detail.

Type 1. Terms of the form $p{\cdot}k'$ that occur in a numerator where k' is not collinear with p, but is approximately collinear with p'. Here we want to understand how to approximate the scalar product by $p{\cdot}p'$.

Type 2. Terms of the form $p_n{\cdot}k'$ that occur in a denominator where k' is approximately collinear with p_n, so the scalar product is very small. Here we want to understand how to approximate the scalar product by $p{\cdot}k'$.

Type 1 approximations. Here we consider the non-vanishing $p{\cdot}k'$

$$p{\cdot}k' = E_p\omega' - |\underline{p}||\underline{k}'|\cos(\theta_{k'})\,, \tag{B.1}$$

where $\theta_{k'}$ is the large angle between the momenta $\underline{p}$ and $\underline{k}'$. Neglecting terms of order m (which would be collinear finite) we can replace here $|\underline{k}'|$ by $|\underline{p}'|\omega'/E_{p'}$. Similarly the angle in $\cos(\theta_{k'})$ may be replaced by $\cos(\theta_{p'})$ since the correction is of the order of the small angle δ and will only introduce finite corrections to our integrals. In this way we see that, for the divergent terms, we may rewrite the numerator using

$$p{\cdot}k' = p{\cdot}p'\frac{\omega'}{E'}\,. \tag{B.2}$$

This manipulation allows us to express several divergent structures via a small number of integrals.

Type 2 approximations. We are interested in approximating $p_n{\cdot}k'$, where p_n and k' are on-shell but p is not. We are interested in the region where p_n is almost collinear with k' so these terms are small. Naively, we might expect that the simple identity

$$p_n{\cdot}k' = (p - nk'){\cdot}k' = p{\cdot}k'\,, \tag{B.3}$$

would suffice. However, in the text, we need to compare $p_n{\cdot}k'$ with $p{\cdot}k'$ where p is an on-shell momentum which is not the case in (B.3) except in the exactly collinear (and hence vanishing) limit. Thus a more careful argument is needed.

To proceed we need to relate the angles between the various vectors. Let θ_n be the small angle between p_n and k', and let θ be the small angle between p and θ. Recall that $p_n = p - nk'$ and hence p is not on-shell. Indeed, it is straightforward to show that for small θ_n we have

$$p^2 = nE_n\omega'\theta_n^2\,. \tag{B.4}$$

Similarly, writing p_n in terms of p and k', we see that the on-shell condition for p_n implies that

$$0 = nE_n\omega'\theta_n^2 - 2nE\omega' + 2n|\underline{p}|\omega'(1 - \tfrac{1}{2}\theta^2)\,. \tag{B.5}$$

Now the spatial component of p can be written as $\underline{p} = \underline{p_n} + n\underline{k}$. Hence we can write

$$|\underline{p}| = E - \frac{nE_n}{2E}\omega'\theta_n^2\,. \tag{B.6}$$

Combining this result with (B.5) we find the approximation

$$E_n^2\theta_n^2 = E^2\theta^2\,. \tag{B.7}$$

Using this result and the small angle approximation that

$$\frac{1}{p_n{\cdot}k'} = \frac{2}{\omega' E_n(\theta_n^2 + \frac{m^2}{E_n^2})}\,, \tag{B.8}$$

it is now straightforward to show that

$$\frac{1}{p_n{\cdot}k'}\frac{E}{E_n} = \frac{1}{p{\cdot}k'}\,. \tag{B.9}$$

C. An example of a dangerous argument

In order to highlight the lack of convergence in summing the degenerate processes that are essential for the Lee-Nauenberg theorem, we will apply to the tree-level process the argument that was used in [15] at higher orders to try to prove infrared finiteness for Coulomb scattering.

We start by considering the tree-level contribution $P_{0,0}$. This is the basic process described by figure 1. At this order in perturbation theory disconnected contributions can only arise due to the introduction of disconnected soft photons. So, for example, in $P_{1,1}$ the disconnected photon lines contract upon themselves resulting in the ill defined volume, denoted by $\Box$, mentioned in section 7. So we can write $P_{1,1} = \Box P_{0,0}$. In a similar manner, the combinatorics of contracting disconnected lines leads to the identity $P_{2,2} = 2(\Box^2 + \Box)P_{0,0}$. In general, we see that the Lee-Nauenberg probabilities that contribute to the "tree-level" cross-section are

$$P_{m,m} = \frac{1}{m!m!}D(m,m)P_{0,0}\,, \tag{C.1}$$

where $D(m,m)$ are the disconnected contributions that arise from m straight through soft photon lines. So as we have seen $D(0,0) = 1$, $D(1,1) := \Box$ and $D(2,2) = 2(\Box^2 + \Box)$. Note that, just as in [15], we are not going to concern ourselves with making sense of $\Box$.

The Lee-Nauenberg total cross-section is then

$$P = \sum_{m=0}^{\infty} P_{m,m} = \left(\sum_{m=0}^{\infty}\frac{1}{m!m!}D(m,m)\right)P_{0,0}\,. \tag{C.2}$$

Now, if we apply the reasoning used in [15] to this, we can derive something which is clearly incorrect.

First note the trivial identity

$$\begin{aligned}\frac{1}{m!m!}D(m,m) &= \sum_{a=0}^{m}\frac{1}{(m-a)!(m-a)!}D(m-a,m-a)\\ &\qquad -\sum_{a=1}^{m}\frac{1}{(m-a)!(m-a)!}D(m-a,m-a)\,.\end{aligned} \tag{C.3}$$

Here we are simply writing a term as a sum minus the same sum without the term. To make sense of this we must have $m > 0$. Now we write this identity as

$$\frac{1}{m!m!}D(m,m) = \sum_{a=0}^{m} \frac{1}{(m-a)!(m-a)!} D(m-a, m-a) \\ - \sum_{a=0}^{m-1} \frac{1}{(m-1-a)!(m-1-a)!} D(m-1-a, m-1-a)\,. \tag{C.4}$$

Hence, using the same argument as in [15], we have

$$\begin{aligned} P &= P_{0,0} + \sum_{m=1}^{\infty} P_{m,m} \\ &= P_{0,0} + \sum_{n=1}^{\infty} \frac{1}{m!m!} D(m,m) P_{0,0} \\ &= P_{0,0} + \sum_{m=1}^{\infty} \sum_{a=0}^{m} \frac{1}{(m-a)!(m-a)!} D(m-a, m-a) P_{0,0} \\ &\quad - \sum_{m=1}^{\infty} \sum_{a=0}^{m-1} \frac{1}{(m-1-a)!(m-1-a)!} D(m-1-a, m-1-a) P_{0,0}\,. \end{aligned} \tag{C.5}$$

Now we can absorb $P_{0,0}$ into the first sum and shift the m counter in the second sum to $m-1$ to get

$$P = \sum_{m=0}^{\infty} \sum_{a=0}^{m} \frac{1}{(m-a)!(m-a)!} D(m-a, m-a) P_{0,0} \\ - \sum_{m=0}^{\infty} \sum_{a=0}^{m} \frac{1}{(m-a)!(m-a)!} D(m-a, m-a) P_{0,0}\,. \tag{C.6}$$

from which we can conclude that

$$P = \sum_{m=0}^{\infty} \sum_{a=0}^{m} \frac{1}{(m-a)!(m-a)!} D(m-a, m-a)(P_{0,0} - P_{0,0}) = 0\,. \tag{C.7}$$

This is precisely the tree-level version of [15, eq. (14)]. The implication, that there is no tree-level scattering, shows that their argument is not safe!

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